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ON A GENERALIZED NOTION OF CUMULANTS

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ABSTRACT. We propose a generalization of the notion of (joint) cumulants, associated to various notions of "independence" in the context of noncommutative probability. This note is mainly based on [6, 7].

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1. WHAT ARE CUMULANTS?

In the usual context of probability theory, the n-th cumulant $k_n(X)$ for a random variable X with all m-th moments $M_m(X) := E(X^m)$ is defined as follows:

$$\exp(\sum_{n=1}^{\infty}\frac{k_n(X)}{n!}t^n) = \sum_{m=0}^{\infty}\frac{M_m(X)}{m!}t^m$$

For example, $k_1(X) = E(X)$ is nothing but the usual expectation of X and $k_2(X) = V(X) := E((X-E(X))^2)$ is called the variance of X. We pick up three essential properties of $k_n(X)$:

(k1) $k_n(\lambda X) = \lambda^n k_n(X)$

(k2) There exists a polynomial P_n such that

$$k_n(X) = M_n(X) + P_n(\{M_p(X)\}_{1\leq p\leq n-1}).$$

(k3) For independent random variables X and Y, $k_n(X+Y) = k_n(X) + k_n(Y)$.

By making use of these properties, you can easily derive the central limit thoerem or Poisson's law of small numbers (at least for the random variables with all finite moments). It shows that cumulants and their properties above play essential role in probability theory.

Moreover, we can define the multivariate version of cumulants, so called "joint cumulants" (multivariate cumulants), which satisfy the following:

(K1) Multilinearity: $K_n : \mathcal{A}^n \to \mathbb{C}$ is multilinear, where $\mathcal{A}$ denotes an algebra of random variables (with all finite moments).

(K2) Polynomiality: There exists a polynomial P_n such that

$$K_n(X_1,\cdots,X_n) = E(X_1\cdots X_n) + P_n(\{E(X_{i_1}\cdots X_{i_p})\}_{\substack{1\leq p\leq n-1,\\ i_1<\cdots<i_p}}).$$

(K3) Vanishment: If $X_1,\cdots,X_n$ are divided into two independent parts, i.e., there exist nonempty, disjoint subsets I, $J\subset\{1,\cdots,n\}$ such that $I\cup J = \{1,\cdots,n\}$ and $\{X_i, i\in I\}$, $\{X_i, i\in J\}$ are independent, then

$$K_n(X_1,\cdots,X_n) = 0.$$

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Covariance $C(X,Y) := E((X-E(X))(Y-E(Y)))$ is an example of joint cumulants ($n=2$ case). As is well known, covariance is useful to evaluate the degree of interdependence between random variables (e.g., $C(X,Y)=0$ if X and Y are independent). In general, we have quantitative evaluation of "independence" by making use of joint cumulants.

In this paper, we propose a generalization of (joint) cumulants associated to "various kinds of independence" in the context of noncommutative probability, which is discussed in the next section.

2. Noncommutative probability

In noncommutative probability theory, we have many kinds of generalized notion of "independence". The essential idea is that a notion of "independence" provides canonical factorization rules for (joint) moments such as $\varphi(X_1...X_n)$.

Let $(\mathcal{A},\varphi)$ be an algebraic probability space, i.e., a pair of a unital $*$-algebra and a state on it. Let $\mathcal{A}_\lambda$ be $*$-subalgebras, where $\lambda \in \Lambda$ are indices. The above mentioned four independences are defined as rules to calculate moments $\varphi(X_1\cdots X_n)$ for

$$X_i \in \mathcal{A}_{\lambda_i},\ \lambda_i \neq \lambda_{i+1},\ 1 \le i \le n-1,\ n \ge 2.$$

Definition 2.1. (1) Tensor independence: $\{\mathcal{A}_\lambda\}$ is tensor independent if

$$\varphi(X_1...X_n) = \prod_{\lambda\in\Lambda} \varphi\Big(\overrightarrow{\prod_{i;X_i\in\mathcal{A}_\lambda}} X_i\Big),$$

where $\overrightarrow{\prod}_{i\in V} X_i$ is the product of $X_i,\ i \in V$ in the same order as they appear in $X_1\cdots X_n$.
(2) Free independence [19]: We assume all $\mathcal{A}_\lambda$ contain the unit of $\mathcal{A}$. $\{\mathcal{A}_\lambda\}$ is free independent if

$$\varphi(X_1...X_n) = 0$$

holds whenever $\varphi(X_1) = ... = \varphi(X_n) = 0$.
(3) Boolean independence [18]: $\{\mathcal{A}_\lambda\}$ is Boolean independent if

$$\varphi(X_1...X_n) = \varphi(X_1)\cdots\varphi(X_n).$$

(4) Monotone independence [10]: We assume that Λ is equipped with a linear order $<$. Then $\{\mathcal{A}_\lambda\}$ is monotone independent if

$$\varphi(X_1...X_i...X_n) = \varphi(X_i)\varphi(X_1...X_{i-1}X_{i+1}...X_n)$$

holds when i satisfies $\lambda_{i-1} < \lambda_i$ and $\lambda_i > \lambda_{i+1}$ (one of the inequalities is eliminated when $i=1$ or $i=n$).

Many probabilistic notions have been introduced for each kind of independence. Analogy of cumulants is a central topic in this field ([19, 20, 16] for free case, [18, 8] for Boolean case).

Lehner [8] unified many kinds of cumulants in noncommutative probability theory in terms of Good's formula. A crucial idea was a very general notion of independence called an exchangeability system . Monotone cumulants however cannot be defined in Lehner's approach. This is because monotone independence is noncommutative: if X and Y are monotone independent, then Y and X are not necessarily monotone independent. Therefore, the concept of "mutual independece of random variables" fails to hold. In spite of this, we found a way to define monotone cumulants uniquely for single variable in [6]. In the present paper, we generalize the method to define joint cumulants for monotone independence.

For tensor, free and Boolean cumulants, the following properties are considered to be basic, as we have discussed for classical case (a special case for tensor case) in introduction.

(K1) Multilinearity: $K_n : \mathcal{A}^n \to \mathbb{C}$ is multilinear.
(K2) Polynomiality: There exists a polynomial P_n such that

$$K_n(X_1, \cdots, X_n) = \varphi(X_1 \cdots X_n) + P_n\big(\{\varphi(X_{i_1} \cdots X_{i_p})\}_{\substack{1 \le p \le n-1, \\ i_1 < \cdots < i_p}}\big).$$

(K3) Vanishment: If $X_1, \cdots, X_n$ are divided into two independent parts, i.e., there exist nonempty, disjoint subsets I, $J \subset \{1, \cdots, n\}$ such that $I \cup J = \{1, \cdots, n\}$ and $\{X_i, i \in I\}$, $\{X_i, i \in J\}$ are independent, then $K_n(X_1, \cdots, X_n) = 0$.

Cumulants for single variable can be defined from joint cumulants: $K_n(X) := K_n(X, \cdots, X)$. Clearly the additivity of cumulants for single variable follows from the property (K3): $K_n(X + Y) = K_n(X) + K_n(Y)$ if X and Y are independent.

The additivity of monotone cumulants for single variable does not hold because of the noncommutativity of monotone independence. Instead, we proved in [6] that monotone cumulants for single variable satisfy that $K_n^M(N.X_1) := K_n^M(X_1 + \cdots + X_N) = NK_n^M(X_1)$ holds if $X_1 \cdots, X_N$ are identically distributed and monotone independent.

The notion of a "dot operation" such as $N.X_1$" is important throughout this paper. This notion was used in the classical umbral calculus [14]. The next section is devoted to the definition of the dot operation associated to each notion of independence.

It enables us to define joint cumulants for natural independence in a unified way, in the section 4, along an idea similar to [6]. The new notion here is monotone joint cumulants denoted as K_n^M. The property (K3) however does not hold for the reason above. Alternatively, it is expected that (K3) holds for identically distributed random variables in view of the single-variable case. This is, however, not the case; as we shall see later, $K_3^M(X, Y, X) \neq 0$ for monotone independent, identically distributed X and Y. To solve this problem, we generalize the condition (K3) in Section 4. We can prove the uniqueness of joint cumulants under the generalized condition. Moreover, we prove the moment-cumulant formulae for the monotone case in Section 5.

3. Dot operation

We used in [6] the dot operation associated to a given notion of independence. This is also crucial in the definition of joint cumulants for natural independence, that is, tensor, free, Boolean and monotone ones.

Definition 3.1. We fix a notion of independence among tensor, free, Boolean and monotone. Let $(\mathcal{A}, \varphi)$ be an algebraic probability space. We take copies $\{X^{(j)}\}_{j \ge 1}$ (in some extended algebraic probability space) for every $X \in \mathcal{A}$ such that
(1) $\varphi(X_1^{(j)} X_2^{(j)} \cdots X_n^{(j)}) = \varphi(X_1 X_2 \cdots X_n)$ for any $X_i \in \mathcal{A}$, $j, n \ge 1$;
(2) the subalgebras $\mathcal{A}^{(j)} := \{X^{(j)}\}_{X \in \mathcal{A}}$, $j \ge 1$ are independent.

Then we define the dot operation $N.X$ by

$$N.X = X^{(1)} + \cdots + X^{(N)}$$

for $X \in \mathcal{A}$ and a natural number $N \geq 0$. We understand that $0.X = 0$. Similarly we can iterate the dot operation more than once; for instance $N.(M.X)$ can be defined (in a suitable space. For details, see [7]).

Remark 3.2. The notation $N.X$ is inspired from "the classical umbral calculus" [14]. Indeed, this notion can be used to develop some kind of umbral calculus in the context of quantum probability.

The power of "dot operation methods" is based on the next proposition;

Proposition 3.3. *(Associativity of dot operation). We fix a notion of independence among the four. Then the dot operation satisfies that*

$$\varphi\big(N.(M.X_1)\cdots N.(M.X_n)\big) = \varphi\big((MN).X_1\cdots(MN).X_n\big)$$

for any $X_i \in \mathcal{A}$, $n \geq 1$.

Proof. $N.(M.X_i)$ is the sum

$$X_i^{(1,1)} + X_i^{(2,1)} + \cdots + X_i^{(M,1)} + X_i^{(1,2)} + \cdots + X_i^{(M,N)}, \tag{3.1}$$

where $\{X_i^{(1,j)}\}_{i=1}^n, \cdots, \{X_i^{(M,j)}\}_{i=1}^n$ are independent for each j and $\{X_i^{(1,j)}+X_i^{(2,j)}+\cdots+X_i^{(M,j)}\}_{i=1}^n$ $(j=1,\cdots,N)$ are independent. On the other hand, $(NM).X_i$ is the sum

$$X_i^{(1)} + \cdots + X_i^{(NM)}, \tag{3.2}$$

where $\{X_i^{(1)}\}_{i=1}^n, \cdots, \{X_i^{(NM)}\}_{i=1}^n$ are independent. Since natural independence is associative, the random variables in (3.2) satisfy a stronger condition of independence than those in (3.1). By the way, the condition of independence in (3.1) is enough to calculate the expectation only by sums and products of joint moments of $X_1, \cdots, X_n$. Therefore, $\varphi\big(N.(M.X_1)\cdots N.(M.X_n)\big)$ must be equal to $\varphi\big((MN).X_1\cdots(MN).X_n\big)$. □

4. Generalized cumulants

The following properties are basic for joint cumulants in tensor, free and Boolean independences.

(K1) Multilinearity: $K_n : \mathcal{A}^n \to \mathbb{C}$ is multilinear.

(K2) Polynomiality: There exists a polynomial P_n such that

$$K_n(X_1,\cdots,X_n) = \varphi(X_1\cdots X_n) + P_n\big(\{\varphi(X_{i_1}\cdots X_{i_p})\}_{\substack{1\leq p\leq n-1, \\ i_1<\cdots<i_p}}\big).$$

(K3) Vanishment: If $X_1,\cdots,X_n$ are divided into two independent parts, i.e., there exist nonempty, disjoint subsets I, $J \subset \{1,\cdots,n\}$ such that $I\cup J = \{1,\cdots,n\}$ and $\{X_i, i\in I\}$, $\{X_i, i \in J\}$ are independent, then $K_n(X_1,\cdots,X_n) = 0$.

Monotone cumulants do not satisfy (K3), even if X_i 's are identically distributed. For instance, $K_3^M(X,Y,X) = \frac{1}{2}(\varphi(X^2)\varphi(Y) - \varphi(X)\varphi(Y)\varphi(X))$ if X and Y are monotone independent (see Example 5.4 in Section 5). Instead we consider the following property.

(K3') Extensivity: $K_n(N.X_1,\cdots,N.X_n) = NK_n(X_1,\cdots,X_n)$.

The terminology of extensivity is taken from the property of Boltzmann entropy.

Remark 4.1. More generally, the following condition is enough to prove the uniqueness of cumulants.

(K3") There exists a polynomial Q_n without a constant or a linear term with respect to N such that

$$K_n(N.X_1, \cdots, N.X_n) = NK_n(X_1, \cdots, X_n) + Q_n(N, \{\varphi(X_{i_1} \cdots X_{i_p})\}_{\substack{1 \le p \le n-1, \\ i_1 < \cdots < i_p}}).$$

There is no change in the proof and we do not consider this condition anymore in this paper.

In the tensor, free and Boolean cases, it is well known that there exist cumulants which satisfy (K1), (K2) and (K3), and hence generalized cumulants exist obviously.

Here we discuss the uniqueness of generalized cumulants for all natural independences, including monotone independence.

Theorem 4.2. *For any one of tensor, free, Boolean and monotone independences, joint cumulants satisfying (K1), (K2) and (K3') are unique.*

Proof. We fix a notion of independence. Let $K_n^{(1)}$ and $K_n^{(2)}$ be two cumulants with possibly different polynomials in the condition (K2). Then $\varphi(N.X_1 \cdots N.X_n)$ is of such a form as

(4.1)

$$\begin{aligned}\varphi(N.X_1 \cdots N.X_n) &= NK_n^{(1)}(X_1, \cdots, X_n) + \\ &\quad N^2 \cdot (\text{a polynomial of } N \text{ and } \{K_p^{(1)}(X_{i_1} \cdots X_{i_p})\}_{\substack{1 \le p \le n-1, \\ i_1 < \cdots < i_p}}) \\ &= NK_n^{(2)}(X_1, \cdots, X_n) + \\ &\quad N^2 \cdot (\text{a polynomial of } N \text{ and } \{K_p^{(2)}(X_{i_1} \cdots X_{i_p})\}_{\substack{1 \le p \le n-1, \\ i_1 < \cdots < i_p}}).\end{aligned}$$

The coefficients of N in the above two lines must be the same. Therefore, $K_n^{(1)} = K_n^{(2)}$. □

The above theorem implies that generalized cumulants coincide with the usual cumulants in tensor, free and Boolean independences since (K3') is weaker than (K3). This is nothing but a new characterization of those cumulants.

The existence of cumulants is not trivial. A key fact is the following.

Proposition 4.3. *For tensor, free, Boolean and monotone independence, $\varphi(N.X_1 \cdots N.X_n)$ is a polynomial of N and $\varphi(X_{i_1} \cdots X_{i_k})$ $(1 \le k \le n, i_1 < \cdots < i_k)$ without a constant term with respect to N.*

Proof. First we notice that there exists a polynomial S_n (depending on the choice of independence) for any $n \ge 1$ such that if $\{X_i\}_{i=1}^n$ and $\{Y_j\}_{j=1}^n$ are independent,

(4.2)

$$\begin{aligned}\varphi((X_1 + Y_1) \cdots (X_n + Y_n)) &= \varphi(X_1 \cdots X_n) + \varphi(Y_1 \cdots Y_n) \\ &\quad + S_n(\{\varphi(X_{i_1} \cdots X_{i_p})\}_{\substack{1 \le p \le n-1, \\ i_1 < \cdots < i_p}}, \{\varphi(Y_{j_1} \cdots Y_{j_q})\}_{\substack{1 \le q \le n-1, \\ j_1 < \cdots < j_q}}).\end{aligned}$$

Let $\{X_i^{(j)}\}_{1 \le i \le n, j \ge 1}$ be copies of $X_1, \cdots, X_n$ appearing in Definition 3.1. We prove the theorem by induction about n. The claim is obvious for $n = 1$ since the expectation is linear. We assume that the claim is the case for $n \le k$. We replace X_i and Y_i in (4.2) by $X_i^{(1)}$ and $X_i^{(2)} + \cdots + X_i^{(L+1)}$, respectively. Then one has

$$\varphi((L+1).X_1 \cdots (L+1).X_{k+1}) - \varphi(L.X_1 \cdots L.X_{k+1})$$
$$= \varphi(X_1 \cdots X_{k+1}) + S_{k+1}\big(\{\varphi(X_{i_1} \cdots X_{i_p})\}_{\substack{1\le p\le k,\\ i_1<\cdots<i_p}}, \{\varphi(L.X_{j_1} \cdots L.X_{j_q})\}_{\substack{1\le q\le k,\\ j_1<\cdots<j_q}}\big),$$

where $1 \le p, q \le k$, $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_q$. The right hand side is a polynomial of L by assumption. Therefore, the sum

$$N\varphi(X_1 \cdots X_{k+1}) + \sum_{L=0}^{N-1} S_{k+1}\big(\{\varphi(X_{i_1} \cdots X_{i_p})\}_{\substack{1\le p\le k,\\ i_1<\cdots<i_p}}, \{\varphi(L.X_{j_1} \cdots L.X_{j_q})\}_{\substack{1\le q\le k,\\ j_1<\cdots<j_q}}\big)$$

is also a polynomial of N without a constant. □

Definition 4.4. We define the n-th monotone (resp. tensor, free, Boolean) cumulant K_n^M (resp. K_n^T, K_n^F, K_n^B) by the coefficient of N in $\varphi(N.X_1 \cdots N.X_n)$ for monotone (resp. tensor, free, Boolean) independence.

It is easy to see that multilinearity (K1) and polynomiality (K2) holds. Extensivity (K3') comes from the associative law of the dot operation, as follows.

Proposition 4.5. *The cumulants $K_n^M, K_n^T, K_n^F, K_n^B$ satisfy the condition (K3').*

Proof. The idea is the same as in [6]. We recall that the dot operation is associative:

$$\varphi(M.(N.X_1) \cdots M.(N.X_n)) = \varphi((MN).X_1 \cdots (MN).X_n).$$

By definition, $\varphi(M.(N.X_1) \cdots M.(N.X_n))$ is of such a form as

$$K_n(N.X_1, \cdots, N.X_n) + M^2 \cdot (\text{a polynomial of } M \text{ and} \{\varphi(N.X_{i_1} \cdots N.X_{i_p})\}_{\substack{1\le p\le n-1,\\ i_1<\cdots<i_p}}).$$

Also by definition $\varphi((MN).X_1 \cdots (MN).X_n)$ is of such a form as

$$MNK_n(X_1, \cdots, X_n) + M^2N^2 \cdot (\text{a polynomial of } MN \text{ and} \{\varphi(X_{i_1} \cdots X_{i_p})\}_{\substack{1\le p\le n-1,\\ i_1<\cdots<i_p}}).$$

The coefficients of M coincide, and hence, (K3') holds. □

Remark 4.6. We know that K^T, K^F and K^B are no other than the usual tensor, free and Boolean cumulants, respectively, because of Theorem 4.2. Therefore, it is obvious that the property (K3) holds. However, we can also prove (K3) directly on the basis of Definition 4.4. See [7].

Corollary 4.7. *For any one of tensor, free and Boolean independences, cumulants satisfying (K1), (K2) and (K3) uniquely exist.*

5. The monotone moment-cumulant formula

We call a subset $V \subset \underline{n}$ a block of interval type if there exist i, j, $1 \le i \le n, 0 \le j \le n - i$ such that $V = \{i, \cdots, i + j\}$. We denote by $IB(n)$ the set of all blocks of interval type. The empty block is assumed not to contained in $IB(n)$. Let V be a subset of $\{1, \cdots, n\}$. We express V as $V = \{k_1, \cdots, k_m\}$ with $k_1 < \cdots < k_m$, $m = |V|$. We collect all $1 \le i \le m+1$ satisfying $k_{i-1} + 1 < k_i$, where $k_0 := 0$ and $k_{m+1} := n + 1$. We label them $i_1, \cdots, i_p$. Let $V_1, \cdots, V_p$ be blocks defined by $V_q := \{k_{i_q - 1} + 1, \cdots, k_{i_q} - 1\}$.

In the above notation, we can prove the following.

Proposition 5.1. *If $\{X_i\}_{i=1}^n$ and $\{Y_j\}_{j=1}^n$ are monotone independent,*

$$\varphi((X_1+Y_1)\cdots(X_n+Y_n)) = \sum_{V\subset \underline{n}} \varphi(X_V)\prod_{j=1}^{p}\varphi(Y_{V_j}). \tag{5.1}$$

Proof. The subsets V_j play roles of choosing positions of Y_i's. Then the claim follows immediately. □

Since $\varphi(N.X_1\cdots N.X_n)$ is a polynomial of N, we can define $\varphi(t.X_1\cdots t.X_n)$ for $t\in\mathbb{R}$. We denote this by $\varphi_t(X_1,\cdots,X_n)$.

Corollary 5.2. *We have the following recurrent differential equations.*
(1) $\frac{d}{dt}\varphi_t(X_1,\cdots,X_n) = \sum_{V\subset\underline{n}, V\neq\emptyset} K^M_{|V|}(X_V)\prod_{j=1}^{p}\varphi_t(X_{V_j})$.
(2) $\frac{d}{dt}\varphi_t(X_1,\cdots,X_n) = \sum_{V\in IB(n)} K^M_{|V|}(X_V)\varphi_t(X_{V^c})$.

Proof. We replace X_i and Y_i in Proposition 5.1 by $N.X_i$ and $(N+M).X_i - N.X_i$ respectively. We notice that $\{N.X_i\}_{i=1}^n$ and $\{(N+M).X_i - N.X_i\}_{i=1}^n$ are monotone independent and that $(N+M).X_i - N.X_i$ is identically distributed to $M.X_i$. We replace N by t and M by s and then the equality

$$\varphi\Big((t+s).X_1\cdots(t+s).X_n\Big) = \sum_{V\subset\underline{n}}\varphi(t.X_V)\prod_{j=1}^{p}\varphi(s.Y_{V_j})$$

holds, where $t.X_E$ means $t.X_{e_1}\cdots t.X_{e_r}$ for a subset $E=\{e_1,\cdots,e_r\}$, $e_1<\cdots<e_r$. The equation (1) follows from the coefficient of t. The coefficient of s appears only when $V^c\in IB(n)$ and therefore we obtain (2) by replacing V^c by V. □

Now we prove the moment-cumulant formula which generalizes the result in [6] for the single-variable case. Let $\mathcal{LP}(n)$ be the set of ordered partitions. An element of $\mathcal{LP}(n)$ is denoted as (π,λ) consisting of $\pi\in\mathcal{P}(n)$ and a linear order of the blocks of π. There are $|\pi|!$ ways to choose λ for each π. We denote by $V>_\lambda W$ if V is larger than W under an order λ.

We introduce a partial order $V\succ W$ for $V,\ W\in\mathcal{NC}(n)$ if there are $i,\ j\in W$ such that $i<k<j$ for all $k\in V$. Visually $V\succ W$ means that V lies in the inner side of W. We define a subset $\mathcal{M}(n)$ of $\mathcal{LP}(n)$ by

$$\mathcal{M}(n) := \{(\pi,\lambda); \pi\in\mathcal{NC}(n),\ \text{if } V,\ W\in\pi \text{ satisfy } V\succ W, \text{ then } V>_\lambda W\}. \tag{5.2}$$

An element of $\mathcal{M}(n)$ is called a monotone partition. The set of monotone partitions was first introduced by Muraki in [11] and later independently found by Lenczewski and Sałapata in [9].

Theorem 5.3. *The moment-cumulant formula is expressed as*

$$\varphi(X_1\cdots X_n) = \sum_{(\pi,\lambda)\in\mathcal{M}(n)}\frac{1}{|\pi|!}K^M_\pi(X_1,\cdots,X_n)$$

Proof. We prove this by induction about n. Assume that

$$\varphi_t(X_1\cdots X_k) = \sum_{(\pi,\lambda)\in\mathcal{M}(k)}\frac{t^{|\pi|}}{|\pi|!}K^M_\pi(X_1,\cdots,X_k).$$

holds for $t\in\mathbb{R}$ and $k\leq n$. We notice that an element in $\mathcal{M}(n)$ can be expressed as $(\pi,\lambda)=(V_1,\cdots,V_{|\pi|})$ with $V_1<\cdots<V_{|\pi|}$. We can use a discussion similar to [5, 6]. (The prototype of the discussion is in [15].) Let $IB(k,m)$ be the subset of

$IB(k)$ defined by $\{V \in IB(k); |V| = m\}$. Let 1_k be the partition $\in \mathcal{P}(k)$ consisting of one block. There is a bijection $f : \mathcal{M}(n+1) \to \left(\bigcup_{k=1}^{n} \mathcal{M}(n+1-k) \times IB(n+1,k)\right) \cup \{1_{n+1}\}$ defined by

$$f : (V_1, \cdots, V_{|\pi|}) \mapsto ((V_1, \cdots, V_{|\pi|-1}), V_{|\pi|}).$$

Therefore, the sum $\sum_{(\pi,\lambda)\in\mathcal{M}(n)}$ can be replaced by $\sum_{V\in IB(n+1)} \sum_{(\sigma,\mu)\in\mathcal{M}(n+1-|V|)}$ and we have

$$\begin{aligned}
\sum_{(\pi,\lambda)\in\mathcal{M}(n+1)} \frac{t^{|\pi|}}{|\pi|!} K_\pi^M(X_1, \cdots, X_n) &= \sum_{V\in IB(n+1)} \sum_{(\sigma,\mu)\in\mathcal{M}(n+1-|V|)} \frac{t^{|\sigma|+1}}{(|\sigma|+1)!} K_\sigma^M(X_{V^c}) K_{|V|}^M(X_V) \\
&= \sum_{V\in IB(n+1)} \int_0^t ds \sum_{(\sigma,\mu)\in\mathcal{M}(n+1-|V|)} \frac{s^{|\sigma|}}{|\sigma|!} K_\sigma^M(X_{V^c}) K_{|V|}^M(X_V) \\
&= \sum_{V\in IB(n+1)} \int_0^t ds \varphi_s(X_{V^c}) K_{|V|}^M(X_V) \\
&= \int_0^t \frac{d}{ds} \varphi_s(X_1 \cdots X_{n+1}) ds \\
&= \varphi_t(X_1 \cdots X_{n+1}).
\end{aligned}$$

We used assumption of induction in the third line and Corollary 5.2 (2) in the fourth line. The claim follows from the case $t = 1$. □

Example 5.4. We show the monotone cumulants until the forth order.

$K_1^M(X_1) = \varphi(X_1),\ K_2^M(X_1, X_2) = \varphi(X_1X_2) - \varphi(X_1)\varphi(X_2),$

$K_3^M(X_1, X_2, X_3) = \varphi(X_1X_2X_3) - \varphi(X_1X_2)\varphi(X_3) - \varphi(X_1)\varphi(X_2X_3) - \frac{1}{2}\varphi(X_1X_3)\varphi(X_2)$
$+ \frac{3}{2}\varphi(X_1)\varphi(X_2)\varphi(X_3),$

$K_4^M(X_1, X_2, X_3, X_4) = \varphi(X_1X_2X_3X_4) - \varphi(X_1X_2X_3)\varphi(X_4) - \frac{1}{2}\varphi(X_1X_3X_4)\varphi(X_2)$
$- \frac{1}{2}\varphi(X_1X_2X_4)\varphi(X_3) - \varphi(X_1)\varphi(X_2X_3X_4) - \varphi(X_1X_2)\varphi(X_3X_4)$
$- \frac{1}{2}\varphi(X_1X_4)\varphi(X_2X_3) + \frac{3}{2}\varphi(X_1X_2)\varphi(X_3)\varphi(X_4) + \frac{2}{3}\varphi(X_1X_4)\varphi(X_2)\varphi(X_3)$
$+ \frac{3}{2}\varphi(X_1)\varphi(X_2)\varphi(X_3X_4) + \frac{1}{2}\varphi(X_1)\varphi(X_2X_4)\varphi(X_3) + \frac{3}{2}\varphi(X_1)\varphi(X_2X_3)\varphi(X_4)$
$+ \frac{1}{2}\varphi(X_1X_3)\varphi(X_2)\varphi(X_4) - \frac{8}{3}\varphi(X_1)\varphi(X_2)\varphi(X_3)\varphi(X_4).$

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